

A proof of Catalan's Convolution formula

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Abstract

We give a new proof of the k -fold convolution of the Catalan numbers. This is done by enumerating a certain class of polygonal dissections called k -in- n dissections. Furthermore, we give a formula for the average number of cycles in a triangulation.

1 Introduction

The Catalan numbers are defined as follows.

Definition 1. For any $n \geq 0$,

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$

For $n < 0$, $C_n = 0$.

The Catalan k -fold convolution formula is due to Catalan.

Theorem 2. [2] Let $1 \leq k \leq n$. Then

$$\sum_{i_1 + \dots + i_k = n} C_{i_1-1} \cdots C_{i_k-1} = \frac{k}{2n-k} \binom{2n-k}{n}. \quad (1)$$

Catalan's original proof [2, 3, 4, 5] uses Lagrange inversion. Gessel and Lacrombe [4] give two proofs which use hypergeometric identities. Tedford [6] exhibits several interpretations of the left-hand side of (1). In this note we use another such interpretation, in terms of dissections of polygons, to give a new proof of Theorem 2. We arrive at this proof using Theorem 5, which enumerates a class of polygonal dissections called k -in- n dissections. As another consequence of this enumeration, in Corollary 7 we give a formula for the average number of cycles in a triangulation.

2 The k -in- n dissections

Definition 3. Let $n \geq 3$ and let $0 \leq k \leq n-3$.

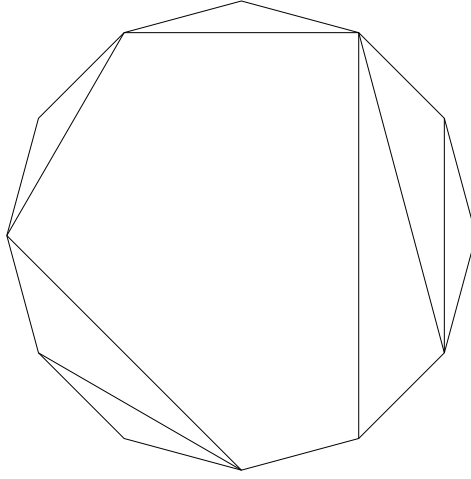


Figure 1: Example of a 5-in-12 dissection

1. A k -dissection of an n -gon is a partition of the n -gon into $k + 1$ parts by k noncrossing diagonals.
2. A triangulation of an n -gon is an $(n - 3)$ -dissection.
3. For $k \geq 4$, an k -in- n dissection is an $(n - k)$ -dissection of an n -gon into one k -gon and $n - k + 1$ triangles (see Figure 1). A 3-in- n dissection is a triangulation with one of its $n - 3$ triangles marked.
4. Let $f_k(n)$ be the number of k -in- n dissections.

It is well known that for $n \geq 3$ the number of triangulations of an n -gon is C_{n-2} .

Lemma 4. *Let $3 \leq k \leq n$. Then*

$$(n - k)f_k(n) = n \sum_{i=2}^{n-k+1} C_{i-1} f_k(n - i + 1). \quad (2)$$

Proof. The left-hand side of (3) is the number of k -in- n dissections, with one of the $n - k$ diagonals marked. These can also be chosen as follows. Choose one vertex v out of the n vertices, then choose $2 \leq i \leq n - k + 1$. Form the diagonal from v to a vertex which is a distance i from v (proceeding, say, counterclockwise along the edges of the n -gon). Mark this diagonal. Now choose a triangulation of the resulting $(i + 1)$ -gon and a k -in- $((n - i) + 1)$ dissection of the resulting $((n - i) + 1)$ -gon. Each such choice results in a unique k -in- n dissection with one of the diagonals marked. \square

Lemma 4 can be used to enumerate the k -in- n dissections.

Theorem 5. *Let $3 \leq k \leq n$. The number of k -in- n dissections is*

$$f_k(n) = \binom{2n - k - 1}{n - 1}. \quad (3)$$

Note 6. There is a bijection between k -in- n dissections and k -crossing partitions of $\{1, \dots, n\}$, as defined in [1]. Thus Theorem 5 is equivalent to [1, Theorem 1].

Theorem 5 implies the following corollary:

Corollary 7. Let $3 \leq k < n$. The average number of cycles of length k in a triangulated n -gon is

$$\binom{2n-k-1}{n-1} \frac{C_{k-2}}{C_{n-2}}.$$

Proof. Each cycle of length k in a triangulation of an n -gon uniquely corresponds to a k -in- n dissection together with a triangulation of a k -gon. The result then follows from (3). \square

The following lemmas will be used in the proof of Theorem 5. It is well known that for any $n \geq 0$,

$$\sum_{i \geq 0} C_i C_{n-i} = C_{n+1}. \quad (4)$$

Lemma 8. For any $n \geq 1$,

$$\sum_{i \geq 0} i C_i C_{n-i} = \binom{2n+1}{n-1}. \quad (5)$$

Proof. Note that

$$\sum_{i \geq 0} i C_i C_{n-i} = \sum_{i \geq 0} (n-i) C_i C_{n-i}.$$

Therefore by (4),

$$\sum_{i \geq 0} i C_i C_{n-i} = \frac{1}{2} \sum_{i \geq 0} n C_i C_{n-i} = \frac{n}{2} C_{n+1} = \binom{2n+1}{n-1}.$$

\square

Lemma 9. Let $1 \leq q \leq p \leq 2q-1$. Then

$$\sum_{i \geq 0} C_i \binom{p-1-2i}{q-1-i} = \binom{p}{q}. \quad (6)$$

Proof. We use induction on q . If $q = 1$ then $p = 1$ and both sides of (6) are equal to 1. Now suppose $q \geq 2$. If $p = q$ then both sides are equal to 1. If $p = 2q-1$ then (6) follows from (4) and (5), since

$$\begin{aligned} \sum_{i \geq 0} C_i \binom{2q-2-2i}{q-1-i} &= \sum_{i \geq 0} C_i (q-i) C_{q-1-i} \\ &= q \sum_{i \geq 0} C_i C_{q-1-i} - \sum_{i \geq 0} i C_i C_{q-1-i} \\ &= q C_q - \binom{2q-1}{q-2} \\ &= \binom{2q-1}{q}. \end{aligned}$$

Now suppose $q+1 \leq p \leq 2q-2$. Note that $q-1 \leq p-1$ and $p-1 \leq 2q-2-1 = 2(q-1)-1$. Therefore by the induction hypothesis, (6) holds for $p-1$ and $q-1$. Also $q \leq p-1$ and $p-1 \leq 2q-3 < 2q-1$, so that (6) holds for $p-1$ and q . Thus

$$\begin{aligned}
\binom{p}{q} &= \binom{p-1}{q-1} + \binom{p-1}{q} \\
&= \sum_{i \geq 0} \frac{1}{i+1} \binom{2i}{i} \binom{p-2-2i}{q-2-i} + \sum_{i \geq 0} \frac{1}{i+1} \binom{2i}{i} \binom{p-2-2i}{q-1-i} \\
&= \sum_{i \geq 0} \frac{1}{i+1} \binom{2i}{i} \binom{p-1-2i}{q-1-i}.
\end{aligned}$$

□

2.1 Proof of Theorem 3

Proof. Fix $k \geq 3$ and proceed by induction on n . If $n = k$ then both sides are equal to 1. Now let $n \geq k+1$. By Lemma 4 and by the induction hypothesis,

$$\begin{aligned}
f_k(n) &= \frac{n}{n-k} \sum_{i=2}^{n-k-1} C_{i-1} f_k(n-i+1) \\
&= \frac{n}{n-k} \sum_{i=2}^{n-k-1} C_{i-1} \binom{2(n-i+1)-k-1}{n-i} \\
&= \frac{n}{n-k} \left(\sum_{i \geq 1} C_{i-1} \binom{2(n-i+1)-k-1}{n-i} - f_k(n) \right).
\end{aligned}$$

Solving for $f_k(n)$ and applying Lemma 9, with $q = n$ and $p = 2n - k$,

$$f_k(n) = \frac{n}{2n-k} \sum_{i \geq 0} C_i \binom{2n-k-2i-1}{n-i-1} = \frac{n}{2n-k} \binom{2n-k}{n} = \binom{2n-k-1}{n-1}.$$

□

3 Proof of the Catalan convolution formula

The next Lemma gives the relation between the number of k -in- n dissections and the Catalan convolution.

Lemma 10. *Let $3 \leq k < n$. Then*

$$k f_k(n) = n \sum_{i_1 + \dots + i_k = n} C_{i_1-1} \cdots C_{i_k-1}. \quad (7)$$

Proof. The left-hand side of (7) is the number of k -in- n dissections, with one of the vertices of the k -gon marked. These can also be chosen as follows. Choose any vertex v of the n -gon. For each vertex v , choose i_1, \dots, i_k such that $i_1 + \dots + i_k = n$. This determines the lengths of the sides of a k -gon by starting at v and proceeding, say, counterclockwise. For example, in Figure 1, if v is the bottom vertex then the lengths are 1, 4, 2, 2, 3. For each $1 \leq r \leq k$, there is a resulting $(i_r + 1)$ -gon sharing one edge of the k -gon. Each of these $(i_r + 1)$ -gon can be triangulated in C_{i_r-1} ways, forming a uniquely determined k -in- n dissection with one of the of the k -gon marked. \square

The proof of Theorem 2 now follows from Lemma 10, since

$$\sum_{i_1+\dots+i_k=n} C_{i_1-1} \cdots C_{i_k-1} = \frac{k}{n} f_k(n) = \frac{k}{n} \binom{2n-k-1}{n-1} = \frac{k}{2n-k} \binom{2n-k}{n}.$$

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